Price is represented by \( p(x) \), small case “\( p \)”; \( x \) represents the number of units that are produced. The price is determined by the demand for the product; items that are small and can be produced easily, such as pencils, are low in price and have a relatively high demand. The reverse is true for items that are complicated to produce and will have a lower demand such as sports cars.

Revenue is the amount of money that is earned from sales; it is represented by \( R(x) \). The revenue is \( \text{the prices times the number of items that are sold} \), so \( R(x) = x(p(x)) \).

The cost of production usually represents a combination of fixed and variable costs. Fixed costs would include, but not be limited to, overhead such as rent or mortgage, security, purchase and depreciation of machinery and some of the utility costs that are not directly involved in the production process. Variable costs would include, but not be limited to, the costs directly involve in the production of the items such as wages, raw materials, utility costs of production, and maintenance. The cost function is represented by \( C(x) \); \( C(x) = \) variable cost + fixed cost. (This is a simplification of the cost function; it may be nonlinear. Think about it, labor costs for increasing production may go up due to overtime pay. Material costs may go down as production increases due to volume purchases of raw materials. As you may recall from Algebra, the equation for a line is \( y = mx + b \); \( m \) is the slope, \( b \) is the \( y \)-intercept.)

Profit, represented by \( P(x) \), is the money derived from subtracting the cost from the revenue.

\[
P(x) = R(x) - C(x)
\]

The word “marginal” refers to the rate of change in the cost, revenue, or profit. When you read or hear the work “marginal”, think “derivative”. The marginal cost, \( MC(x) \), is the derivative of the cost function, \( C'(x) \), marginal revenue, represented by \( MR(x) \), is also the derivative \( R'(x) \), and marginal profit is \( MP(x) \), which is the derivative \( P'(x) \).

Let’s do an example. We’ll want to answer the following questions. What volume of sales, \( x \), will maximize revenue, that is, create the most revenue? What is the maximum revenue? What volume of sales, \( x \), will maximize profit? What is the maximum profit? The price-demand equation for an item is \( x = $5000 - 20p \), that is, according to the price you can get for the item, the number of items that should be produced should be “$5000 minus 20 times the price”. Think about it, as the price increases, the demand will go down since, usually, less people will be willing to pay the higher price.

Find \( p(x) \), that is, the price as a function of \( x \), the number produced. \( p(x) = 0.05x^2 + 250x \) for “\( p \)”. Answers are below.

The revenue function is \( R(x) = x(p(x)) \) MR(x) or \( R'(x) = \) ________________ Take the derivative of \( R(x) \).

What production level will maximize revenue? The revenue will be maximized when the slope of the tangent line to the revenue function is horizontal. That will be when \( R'(x) = 0 \) or \( 0.1x - 250 = 0 \). Show your work.

\[
x = 2500
\]

What is the maximum revenue? Put your \( x \) value into the revenue function, \( R(x) \), not \( R'(x) \), to get the answer.

\[
The maximum revenue is \$312,500
\]

Now let’s do that on the calculator. Put \( R(x) \) into your “\( Y=\)” menu. \( Y_1 = -0.05x^2 + 250x \) Press ZOOM 6; do you see anything? You probably do not; that’s because you have to set your window at an appropriate setting. This is done by changing the \( X_{\text{max}} \) and \( Y_{\text{max}} \), usually by trial and error. Press WINDOW; use the following: -10, 6000, 1, -10, 400000, 1, 1 and press GRAPH. You should see the graph (a parabola). To get the maximum, press “\( 2^{\text{nd}} \) TRACE and choose “4”. Use your arrow key to put the curser anywhere to the left of the maximum point and press ENTER; now move the curser anywhere to the right of the max and press ENTER. You probably don’t have a guess, so press ENTER again and the calculator will give you the coordinates of the maximum: \( x = \) production level of 2500 and \( y = \) the maximum possible revenue of \$312,500.

\[
p(x) = (x - 5000)^{-2}/2 \text{ or } -0.05x + 250; \quad R(x) = x(p(x)) = -0.05x^2 + 250x; \quad MR(x) = R'(x) = -0.1x + 250; \quad R'(x) = \text{zero at } x = 2500
\]

\( x = 2500 \) items will maximize revenue. The max revenue is \( R(2500) = \$312,500 \).
Continuing with the same problem, let’s say the cost function is \( C(x) = 65x + 70000 \) (the cost of producing \( x \) items is $65 each plus the fixed cost of $70,000). What production level, \( x \), would give the “break-even” point? That would be where revenue = the cost: \( R(x) = C(x) \); setting them equal, that would be about \( x = 427 \) or 428 items using the quadratic formula: \(-0.05x^2 + 250x = 65x + 70000 \) or \(-0.05x^2 + 185x - 70000 = 0\). This can also be done on the calculator. Put \( Y_2 = C(x) = 65x + 70000 \) into your graph menu. You already have \( R(x) \) in it. \( Y_1 = -0.05x^2 + 250x \) and \( Y_3 = 65x + 70000 \); press GRAPH since you already have the correct WINDOW. Notice the two intersect at the beginning of the graph. Before that, the cost is higher than the revenue – you’re operating at a loss. After that, revenue is higher than the cost – you’re operating at a profit. (Also notice that costs are greater than revenue if you produce too many items, according to this model/example.)

With both graphs on your screen, press 2nd, TRACE; choose “intersect”; using the arrow key, place the cursor to the left of the intersection and press ENTER. Now move the cursor to the right of the intersection point and press ENTER. Since you don’t have a “GUESS”, press ENTER again and you should get \( x \) is approximately 427.85373. At \( x = \) about 428 units, you are at “break even”, the revenue is equal to the cost; your profit is zero.

**Profit is revenue minus cost.**

The profit function is \( P(x) = R(x) - C(x) \)  
\[
\text{MP}(x) = P'(x) = \frac{R'(x) - C'(x)}{R(x) - C(x)} 
\]

What production level will maximize profit? The profit will be maximized when the slope of the tangent line to the revenue function is horizontal. That will be when \( P'(x) = 0 \) or \( 0.1x - 250 = 0 \). Show your work.  
\[
x = \underline{428} 
\]

What is the maximum profit? Put your \( x \) value into the profit function, \( P(x) \), not \( P'(x) \), to get the answer.

The maximum profit is $________________________

Now let’s do that on the calculator. Put \( P(x) \) into your “Y = “ menu. \( Y_3 = -0.05x^2 + 250x - 65x - 70000 \) and press GRAPH. You should see all three graphs. You should have all three functions in your graph menu: \( Y_1 = -0.05x^2 + 250x \); \( Y_2 = 65x + 70000 \), and \( Y_3 = -0.05x^2 + 250x - 65x - 70000 \) To get the maximum, press “2nd” TRACE and choose “4”; make sure you are on the correct graph by using your “up” or “down” arrow keys. Put the cursor anywhere to the left of the maximum point and press ENTER; now move the cursor anywhere to the right of the max and press ENTER. You probably don’t have a guess, so press ENTER again and the calculator will give you the coordinates of the maximum: \( x = \) production level of 1850 and \( y = \) the maximum possible profit of $101,125. Also, pressing “2nd” TRACE and choosing “zero” will give you the production level, \( x \), of the break-even point. Move the cursor to the left until you get a negative profit and press ENTER; now move the cursor to the right until you get a positive profit; press ENTER twice. You should get 428 as the break-even point. (This process is called “optimization” and is discussed later in this handout.)

Please take time to try to understand the following statements.

Look at the graph of all three functions. This all seems reasonable; you should reach the production level, \( x \), of maximum profit before you reach the production level, \( x \), of maximum revenue since you have to subtract the cost from the revenue in order to get the profit.

The intersection of the revenue function and the cost function is the break-even point. It is the same production level, \( x \), as the x-intercept or “zero” of the profit function since \( \text{profit} = \text{revenue} - \text{cost} \). The physical height of any point on the profit function is equal to the physical height of the revenue function minus the physical height of the cost function.

So, if you are doing on-line homework or test problems and are given figures to choose which are suppose to model a revenue-cost-profit situation, the revenue must be higher than the profit functions and the break-even point must have the same x-value as the x-intercept of the profit function since, at break even, profit is zero.

\[
P(x) = -0.05x^2 + 250x - 65x - 70000; \text{MP}(x) = P'(x) = -0.1x + 185; P'(x) = 0 \text{ at } x = 1850; x = 1850 \text{ items will maximize profit. The max profit is } P(1850) = $101,125.
\]
Using the same examples, find the following.

1. Is revenue increasing or decreasing at \( x = 2000 \)? At what rate is the revenue increasing (or decreasing) when the production level, \( x \), is 2000 units?

2. Is revenue increasing or decreasing at \( x = 2200 \)? At what rate is the revenue increasing (or decreasing) when the production level, \( x \), is 2200 units? How does this rate compare to the increase or decrease at \( x = 2000 \) units?

3. At what rate is the revenue increasing (or decreasing) when the production level, \( x \), is 2600 units? What causes this answer to be what it is?

4. Is profit increasing or decreasing at \( x = 1800 \)? At what rate is the profit increasing (or decreasing) when the production level, \( x \), is 1800 units?

The word “rate” is telling you that the answer is the derivative, the slope of the tangent line to the curve at that \( x \) value. Find the derivative of the revenue function at that specific \( x \) value. (1) \( R(x) = -0.05x^2 + 250x \), so \( MR(x) = R'(x) = -0.1x + 250 \). \( R'(2000) = -0.1(2000) + 250 = 50 \). At the production level of \( x = 2000 \) units, revenue is rising at a rate of $50 per unit.

This can be done on the calculator! You should have the revenue function in the calculator: \( Y_1 = -0.05x^2 + 250x \)

Press 2nd, TRACE and choose #6 - \( \frac{\delta y}{\delta x} \), type in 2000 and press ENTER. You should get “50”.

Please do the rest of the problems.

1. Revenue is increasing; the rate of increase is $50 per unit of production; increasing.
2. Revenue is increasing; \( R'(2200) = 30 \) or revenue is increasing at $30 per unit when production is 2200 unit. The rate at which revenue is increasing is lower at 2200 units than at 2000 units since the cost of production is increasing.
3. \( R'(2600) = -10 \), that is, revenue is decreasing by $10 per unit when 2600 units are being produced. Costs are increasing at a faster rate than revenue is increasing at the 2600 unit-production level.
4. Profit is increasing; \( MP(1800) = P'(1800) = 5 \), that is, profit is increasing at a rate of $5 per unit when production is 1800 units.
Differentials:

1. If production increases from 1800 units to 1810 units, what is the increase (change) in revenue?

2. If production increases from 1800 units to 1810 units, what is the increase (change) in profit?

\[ MR(x) = R'(x) = -0.1x + 250 \]

\[ \delta R \approx \delta x \]

What is \( \delta R \) when \( x = 1800 \) and \( \delta x = 10 \)?

\(-0.1(1800) + 250)(10) = 70(10) = 700.\]

This means that if production is increased by 10 units when it is at 1800 units, the revenue will increase by $700. Using the calculator: Press 2nd, TRACE and choose #6, \( \frac{dy}{dx} \), type in 1800 and press ENTER. You should get “70”. 70(10) = 700. Do #2 showing your method.

2. \[ MP(x) = P'(x) = -0.1x + 185 \]

\[ \delta P \approx \delta x \]

When \( x = 1800 \) and \( \delta x = 10 \), \( \delta P = 5(10) = 50 \), which means if production increases by 10 units when it is 1800 units, the profit will increase by $50.

“Continuous” Compounding – See the handout on logarithms and exponential growth and decay models at http://www.hacc.edu Choose “Lancaster Campus”. Scroll to bottom and choose “Tutoring” under the “STUDENTS” heading. Choose Lancaster Tutoring” on the left side. Scroll down to “Looking for help with your TI 83 or 84 graphing calculator” – in red letters - click on it. Scroll down to ”Handouts”.

Elasticity of Demand: You know that an increase in price will cause a decrease in demand. Even though a price increases, that does not necessarily mean the revenue will decrease. The increases in price might cause a few buyers to leave the market and not buy the produce, but the number that chooses to leave the market place might not be large enough to cause a decrease in revenue; the price increase compensates for the decrease in buyers and raises revenue. Of course, a price increase could certainly cause a decrease in revenue because too many buyers stop purchasing the product. The focus is on price – p(x) Small p stands for “price” and capital P stands for “profit”.

Elastic Demand: An increase in price causes a decrease in revenue; a price decrease causes a revenue increase.

Inelastic Demand: A price increase causes an increase in revenue; a price decrease price causes a revenue decrease.

If “E(p)” represent “elasticity of demand”; it is the negative of “p” times f ‘(p) divided by f(p):

\[ E(p) = -\frac{pf'(p)}{f(p)} \]

If E(p) is between 0 and 1, then demand is not sensitive to a change in price; it is inelastic. A change in price causes a smaller change in demand.

If E(p) is greater than 1, then demand is sensitive to a change in price; it is elastic; a change in price causes a large change in demand.

According to your book, elasticity, E(p), is a measure of how much the demand changes for a given change in price.”

Using the previous price-demand function of \( x = f(p) = 5000 - 20p \), so \( f'(p) = -20 \). Change the function to \( f(p) = 20(250-p) \), this will be helpful when you find the formula for E(p).

\[ E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p(-20)}{20(250-p)} = -\frac{p}{250-p} \]
What is $E(50)$? That would be $\frac{50}{250 - 50} = \frac{1}{4}$, which is $< 1$. At price $p = 50, the demand is inelastic. If the price of $50$ changes by 10%, then the demand will change by approximately $50(0.10) = 5\%$.

Find $E(130)$ showing your work. Is demand elastic or inelastic at a price of $130$?

$E(130) = \frac{130}{250 - 130} = \frac{130}{120} \approx 1.08$, which is $> 1$. At $p = 130$, the demand is elastic. Note: If the price of $130$ changes by 5%, then the demand will change by approximately $130(0.05) = 6.5\%$.

**Optimization:** Examples of optimization problems would include finding the maximum or minimum of a curve, such as the production level for producing optimal revenue or profit, as previously demonstrated on page 2. For the new optimization problems, you are given facts to form two equations, a constraint and an optimization equation. These need to be “blended together” to get one equation in one unknown. Let’s look at some examples.

Example 1: Parcel delivery services constrain the size of the package to a length and “girth” of 108 inches; girth is the distance around it. If a package has square ends, what is the maximum volume it can have? The constraint equation is $l + w + h = 108$, but the width and height are the same, so you get $108 = l + 2w$. The optimization equation is $V = lwh$, length times width times height or $V = lw^2$. Let’s solve for “l” in the constraint equation to substitute it into the optimization equation. $1 = 108 - 2w$. $V = lw^2 = (108 - 2w)w^2 = 108w^2 - 2w^3$. Since you want to maximize the volume, let’s take the derivative and set it equal to zero: $\frac{\delta V}{\delta w} = V'(w) = 216w - 6w^2 = 0$ or $w = 36$ or $w = 36$ inches.

That makes the length $= 108 - 2(36) = 36$ inches. It’s a cube! The maximum volume is $36^3 = 46,656$ cubic inches.

Now let’s use the calculator. Put the optimization equation in your calculator: $Y_1 = (108 - 2x)x^2$ using the WINDOW -10, 60, 1, -10, 50000, 1, 1; press GRAPH and 2nd Trace and choose maximum. For your left bound, put the curser to the left of the maximum, press ENTER, for the right bound, put the curser to the right of the max, press ENTER; since you may not have a guess, press ENTER. You should get $x = 36$ and the maximum volume, $y_1 = 46656$.

Example 2: A large hotel in a large city has a capacity of 300 rooms. It is usually filled to capacity when customers are charged $90 per night. For each increase of $2 per room, it is projected, through a market study, that the hotel will lose 5 customers. What number of price increases will maximize revenue? Let’s start with a table in order to get an understanding of what is happening. “$n$” = the number of price increases.

<table>
<thead>
<tr>
<th>$x$ = prices increases</th>
<th>$x$ price/room</th>
<th>rooms rented</th>
<th>revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$90$</td>
<td>300</td>
<td>$27,000$</td>
</tr>
<tr>
<td>1</td>
<td>92</td>
<td>295</td>
<td>$27,140$</td>
</tr>
<tr>
<td>2</td>
<td>94</td>
<td>290</td>
<td>$27,260$</td>
</tr>
<tr>
<td>3</td>
<td>96</td>
<td>285</td>
<td>$27,360$</td>
</tr>
<tr>
<td>$n$</td>
<td>$90 + 2n$</td>
<td>$300 - 5n$</td>
<td>$(90 + 2n)(300 - 5n)$</td>
</tr>
</tbody>
</table>
You want to maximize revenue, so find the derivative of the revenue function and set it equal to zero.

\[ R(n) = (90 + 2n)(300 - 5n) = 10n^2 + 150n + 27000 \quad \text{MR}(n) = R'(n) = -20n + 150; \text{MR}(n) = 0 \text{ at } n = 7.5. \]

Revenue will be maximized (“optimal” revenue) at \( n = 7 \) or 8 price increases of $2 per increase. The price per room should be $90 + 2(7) or $104.

What is the maximum revenue for the price of $104 per room? _______________

Let’s do this on the calculator; put the revenue function into the calculator: \( Y_1 = (90 + 2x)(300 - 5x) \). Play with the WINDOW until you can see it. Press 2nd TRACE and choose “maximum”; put in an \( x \) value left of the maximum, press ENTER; put in an \( x \) value right of the max and press ENTER. Press ENTER again. What did you get?

Now press TRACE and enter the number 7; what revenue would you get? ________________

If you TRACE on 8, do you get the same revenue? ______________

$27,560

Inventory Control: A business sells 160,000 items uniformly throughout the year, that is, this item is not “seasonal”. It costs $8 to store this item for the year. The cost of ordering more of them is $100 per order. The cost of replenishing the inventory, buying \( x \) items, is \( C(x) = \text{(cost of storage)} \left( \frac{x}{2} \right) + \text{(cost of ordering)}(\text{number of times ordered}) \).

“Average number stored” is going to be the number ordered each time, \( x \), divided by 2; on average, at any time, half have been sold and half are still in the store.

\[ C(x) = 8 \left( \frac{x}{2} \right) + 100 \frac{160000}{x} \]

where \( r \) = number of times orders are made per year.

Since the business sells 160,000 items per year, \( r = \frac{160000}{x} \), so \( C(x) = 8 \left( \frac{x}{2} \right) + 100 \frac{160000}{x} \) or

\[ C(x) = 8 \left( \frac{x}{2} \right) + 16,000,000 x^{-1} \quad \text{Costs are minimized when the derivative, } C'(x) = 0. \]

\[ C'(x) = 4 - 16,000,000 x^{-2} \text{ or } 4 - \frac{16,000,000}{x^2} = 0 \text{ at } x^2 = 4,000,000 \]

Costs are minimized when \( x = 2000 \).

Future Value of a Continuous Income Stream (Investment): The future value of a continuous income stream is the continuous income stream (income and interest) at the end of \( t \) years. This assumes the income is continuously invested at a rate that is compounded continuously; an example might be a retirement account that continually gets the same rate of interest, even though this is somewhat unrealistic. Remember that continuous growth or decay is modeled by \( e^x \). The formula is \( e^{rt} \int_0^t f(t) e^{-rt} \, dt \).

If the “flow rate”, \( f(t) \), is a set value, such as investing $2000 per year, it is just used in place of \( f(t) \); if \( f(t) \) is given as a formula, it must be integrated and be a part of the integrand.

Example: \( t = 10 \text{ years}, r = 6\% \), \( f(t) = $1500 \text{ per year} \)

Remember that 6% is a reasonable return on something like diversified mutual funds over a long time. The future value is \( e^{0.06(10)} \int_0^{10} 1500 e^{-0.06t} \, dt \). Let’s do this in the calculator. Press MATH, choose #9, and put in the following: \( \text{fnInt}( e^{0.06(10)}(1500 e^{-0.06x}), x, 0, 10) = \text{fnInt}(e^{(0.06*10)}(1500 e^{-0.06x}), x, 0, 10) \) and press ENTER. What did you get? ________________

Does this seem reasonable? What is $1500(10 years)? Don’t you expect something larger than $15000? (Note: Your TI-84 may have an integration sign when you press MATH #9.)

Example: \( t = 10 \text{ years}, r = 6\% \), \( f(t) = 4000 \text{ per year} \)

The future value is \( \text{fnInt}(e^{(0.06*10})(4000 e^{(-0.06x)}), x, 0, 10) = \text{________} \) Does this seem reasonable? Saving $4000 for 10 years would be $40,000; now add the interest.

$20552.97; $54807.92

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