How does the “shell” method work for finding volumes of figures? What is the volume of the function \( f(x) = x^2 + 1 \) rotated about the y-axis on the interval \([0, 3]\), that is, \(0 \leq x \leq 3\).

What is the volume of a roll of paper towel that is 12 inches high and has a radius of 5 inches? It has a core that is 1 in in radius. Wouldn’t that be \( V = \pi R^2 h - \pi r^2 h \) where \( R \) is the radius of the outside and \( r \) is the radius of the core?

\[
V = 12 \pi (5^2 - 1^2) = 288 \pi \approx 904.779
\]

Now think of the roll of paper towel as a stack of rectangular sheets (shells) that are wrapped around the core. Each sheet is a rectangle that is 12 inches high and has a radius that is changing from 1 inch to 5 inches; each sheet is extremely thin. The length of each rectangle is the circumference of the “shell”.

You want to add all these thin sheets (areas) together to get the volume. Integration is a “summation” process.

Now picture the line \( f(x) = y = 12 \) representing the height. The center of the core is the y-axis. The radius is “x” and it is changing from \( x = 1 \) to \( x = 5 \); the “thickness” is \( \delta x \). The change in radius is represented by \( \int_1^5 x \, \delta x \); the height is the function; in this case, \( f(x) = 12 \).

The volume is \( 2\pi \int_1^5 12x \, \delta x \) = the sum of the areas; let’s see if it’s equal to \( 288 \pi \).

We’ll do it on the calculator first. At your home screen, press MATH, choose #9: “fnInt(“ means finite integral. Type in “24x, x, 1, 5”; it should look like this: fnInt(24x, x, 1, 5). Press ENTER; what did you get? \( _____ \) times \( \pi = _____ \) ALL RIGHT! Notice \( 2\pi \int_1^5 12x \, \delta x = 2\pi (12 \left( \frac{x^2}{2} \right) ) \bigg|_1^5 = 12 \pi (25 - 1) = 288 \pi \)

Now let’s try this problem. Find the volume of the region bounded by the function \( f(x) = x^2 + 1 \), \( y = 0 \) and \( x = 3 \) that is rotated about the y-axis. The object is going to look like a “cup” with a very thick bottom. Notice the “shells” or sheets of very thin “paper” have a radius starting at zero and going to 3 units. The heights of each shell is the function, \( x^2 + 1 \); the height is going from 1 (\( f(0) - 0 = 1 - 0 = 1 \)) to 10 (\( f(3) - 0 = 10 - 0 = 10 \)). The volume is found by adding these shells together. Here’s what it looks like.

\[
V = 2 \pi \int_0^3 x(x^2 + 1) \, \delta x ; \text{ remember, the radius of each shell is } \int_0^3 x \, \delta x ; \text{ the radius starts at 0 and increases to}
\]

Doing it on the calculator, (press MATH, choose #9) it’s \( 2\pi \) fnInt( \( x(x^2 + 1) \), \( x, 0, 3 \)). What did you get? \( _____ \) I got \( \approx 155.509 \) or the exact answer of \( 49.5 \pi \). (The TI-84 uses a different symbol.) Doing it “synthetically” using “algebra” and the formula,

\[
V = 2 \pi \int_0^3 x(x^2 + 1) \, \delta x = 2 \pi \int_0^3 (x^3 + x) \, \delta x = 2 \pi \left( \frac{x^4}{4} + \frac{x^2}{2} \right) \bigg|_0^3 = 2 \pi \left( (\frac{81}{4} + \frac{9}{2}) - 0 \right) = 2 \pi (20.25 + 4.5) = 49.5 \pi
\]
What's the volume of a cone that is 8 inches high and has a radius of 4 at the base? $V = \frac{1}{3} \pi r^2 h = \pi \frac{1}{3}(4^2)(8) = 42 \frac{2}{3} \pi \approx 134.041$ Now think of the line starting at (0,0) and ending at (8, 4) being rotated around the x-axis between x = 0 and x = 8. That would generate a cone. You could think of this cone’s volume as a stack of extremely thin circles stacked on top of each other, each circle having a radius that is slightly smaller than the previous one. The radii would go from a radius of 4 inches to zero inches. Each circle would have a thickness of almost nothing, $\delta x$, making it a “disc”. (If the solid was a cylinder inside a cylinder, that stack of circles / discs would become “washers”.) Let’s examine this situation.

We need to “add” the area of these discs together to get the volume; integration is a “summation” process. The thickness of each disc is represented by $\delta x$ and the change in the size of the radii is in the change in the indices of the integral. The radii themselves are the functional values, f(x) or “y” values, as x changes from 0 to 8. What is the linear function that contains (0,0) and (8,4)?

The line function is $f(x) = \frac{0.5x}{x}$. Using the information, the volume of the cone is the sum of the areas of each disc; the area of a circle is $\pi r^2$, the radius is represented by f(x), so the volume is

$$V = \int_0^8 (0.5x)^2 \delta x.$$ Let’s check this on the calculator. Press MATH, choose #9 - “fnInt (“ means finite intergral.

You should have the following: fnInt( (0.5 x^2 ), x, 0, 8), ENTER. What did you get? ________________ Now take that times $\pi$ and you should get $\approx 134.041$. The exact answer: $128 \frac{\pi}{3}$; 134.041 is an approximate answer.

Now let’s do a more complicated problem. The region bounded by the function $f(x) = x^2 + 1$, and $y = 0$, is rotated about the y-axis between x = 0 and 3. What is the volume of the figure? The object is going to look like a “cup” with a very thick bottom. Notice the bottom will be “discs” and the top will be “washers”; we will be adding these together relative to the y-axis or by stacking them. The radii will be the distances from the vertical line x = 3 and the curve x as a function of y. Since $y = x^2 + 1$, then $x = \sqrt{y-1}$.

The volume is found by adding these discs and washers together. Here’s what it looks like. Take a little time to think about what was just typed.

Notice that this must be done in two parts since the function does not become part of the problem until $y = 1$.

The sum of the discs will be $\pi \int_0^1 3^2 \delta x$, which is just the volume of a cylinder with radius 3 and height 1, which is $\pi (3^2)(1)$, which is $9 \pi$. Notice the integral $\int_0^1 3^2 \delta x$ is also 9, so $\pi \int_0^1 3^2 \delta x = 9 \pi$ is the volume of the discs.

Recall the area of the washer is $\pi (R^2 - r^2)$, not $\pi (R - r)^2$. We need to know this for the integral for the top part, the washers. Since $f(3) = 3^2 + 1$ or 10, the radii will be $3 - \sqrt{y-1}$ from $y = 1$ to $y = 10$; let’s check that: If $y = 1$, the radius of the washer is 3 and if $y = 10$, the radius of the washer part is 0. That makes sense, so my set-up appears to be correct. The volume of the top or washer part is $\pi \int_1^{10} 3^2 - \sqrt{y-1}^2 \delta y$; that’s $\pi$ fnInt( $(3^2 - \sqrt{x-1}^2$, x, 1, 10), which is $40.5 \pi$; $40.5 \pi + 9 \pi = 49.5 \pi$, the same as we got on the previous sheet using the shell method! **TWO METHODS ARE NEEDED BECAUSE SOME SOLIDS CANNOT BE FOUND USING ONE OF THE METHODS!** A calculator is needed because some functions cannot be integrated, a good approximation is used.